

Holographic Entanglement Entropy of Anisotropic Minimal Surfaces in LLM Geometries

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Abstract

We calculate the holographic entanglement entropy (HEE) of the \mathbb{Z}_k orbifold of Lin-Lunin-Maldacena (LLM) geometries which are dual to the vacua of the mass-deformed ABJM theory with Chern-Simons level k . By solving the partial differential equations analytically, we obtain the HEEs for all LLM solutions with arbitrary M2 charge and k up to μ_0^2 -order where μ_0 is the mass parameter. The renormalized entanglement entropies are all monotonically decreasing near the UV fixed point in accordance with the F -theorem. Except the multiplication factor and to all orders in μ_0 , they are independent of the overall scaling of Young diagrams which characterize LLM geometries. Therefore we can classify the HEEs of LLM geometries with \mathbb{Z}_k orbifold in terms of the shape of Young diagrams modulo overall size. HEE of each family is a pure number independent of the 't Hooft coupling constant except the overall multiplication factor. We extend our analysis to obtain HEE analytically to μ_0^4 -order for the symmetric droplet case.

1 Introduction

Gauge/gravity duality has been a central paradigm for decades in theoretical physics. Among others, holographic calculation of the entanglement entropy (EE) [1, 2] draws recently much attention due to its elegance and implications for the nature of quantum field theories as well as quantum gravity.

In this paper, we consider \mathbb{Z}_k orbifolds of Lin-Lunin-Maldacena (LLM) geometries [3, 4] with $\text{SO}(2,1) \times \text{SO}(4) \times \text{SO}(4)$ isometry in 11-dimensional supergravity and calculate the holographic entanglement entropy (HEE) to nontrivial orders in the mass parameter. The main motivation is their connection to the Aharony-Bergman-Jafferis-Maldacena (ABJM) theory with level k [5] which is a conformal field theory (CFT) describing the dynamics of M2-branes on the transverse $\mathbb{C}^4/\mathbb{Z}_k$ orbifold with the Chern-Simons level k . It allows a mass-deformation [6, 7] which preserves full $\mathcal{N} = 6$ supersymmetries. This mass-deformed ABJM (mABJM) theory has many supersymmetric vacua. It has been shown that the vacua have one-to-one correspondence with the \mathbb{Z}_k orbifold [8, 9] of LLM geometries, which are classified by a 1-dimensional droplet picture, or equivalently Young diagrams [4].

The LLM metric has a mass parameter μ_0 which is proportional to the mass parameter μ in the mABJM theory. Then we can explore the renormalization group (RG) flow of the renormalized entanglement entropy (REE) [10] triggered by the mass deformation from the ABJM theory as a UV fixed point [11]. Since there are many vacua in the theory, the RG flow depends on the vacuum. This should be manifested in the holographic calculation of REE for LLM geometries. See also [12–18] for the behavior of EE under relevant perturbations from the UV fixed point.

An important issue related to REE is about the c -theorem which states that there exists a c -function which is positive definite and monotonically decreasing along the RG flow [19–21]. In 3-dimensions in particular, it is called the F -theorem [22] because the free energy on a three sphere plays the role of c -function. The F -theorem was proved [23] through the connection of the free energy with the constant term of EE of a circle [24–26]. In this paper, we will examine explicitly how F -theorem is realized in the HEE of the mABJM theory which has a large number of discrete vacua.

The LLM geometries with \mathbb{Z}_k orbifold are all asymptotic to $\text{AdS}_4 \times S^7/\mathbb{Z}_k$. They are, however, not spherically symmetric along the radial direction of the AdS geometry but depend on two transverse coordinates. Therefore, it is not a simple exercise to get the minimal area for a given entangling region because one has to solve a partial differential equation (PDE) for the two transverse coordinates. In the previous work [11], the angle

dependence was neglected to simplify the calculation with the assumption that it would not contribute at least in the leading order in μ_0 . Though sensible results were obtained for simple droplet configurations, there were cases that the F -theorem is violated in this approximation. In this work, however, we take into account all the angle dependence exactly. In other words, we solve the PDE exactly for all LLM solutions with \mathbb{Z}_k orbifold and obtain the corresponding HEE up to μ_0^2 -order. Then we verify that the REE satisfies the F -theorem for all relevant deformations connected to dual LLM geometries with \mathbb{Z}_k orbifold. For some simple droplet configurations with general k , we further extend our analysis to μ_0^4 -order and obtain REE analytically.

Since we work with the most general k and the rank of the gauge group N , it is possible to investigate the dependence of EE on these parameters including the 't Hooft coupling $\lambda = N/k$ in particular. Note, however, that it is not a trivial task to compare the EEs with different N or k because they will not uniquely specify a droplet due to many degeneracies. That is, in the field theory language, different vacua will give different EEs and to begin with one has to specify the vacua to compare. We will see that, depending on which vacua to choose, the EE depends on λ differently. Moreover, we will show that, up to a multiplication factor, the HEE of LLM geometries is independent of the overall scaling of the droplet configurations to all orders in μ_0 . Therefore, we can classify the LLM geometries with \mathbb{Z}_k orbifold in terms of the shape of the corresponding Young diagrams modulo overall size. At each order in μ_0 , they are pure numbers independent of λ . These can be considered as nontrivial results to test the gauge/gravity duality in the large N limit between the LLM geometry and mABJM theory which are not conformal.

This paper is organized as follows. In section 2, we briefly review the relation between the vacua of the mABJM theory and the droplet classification of the LLM geometry with \mathbb{Z}_k orbifold. In section 3, we solve the PDE exactly to obtain the HEE of a disk up to μ_0^2 -order. We show that the resulting REE satisfies the F -theorem near the UV fixed point for all LLM geometries with \mathbb{Z}_k orbifold. Then we show that it is classified by the shape of the Young diagrams and discuss how it depends on N and k . We also calculate the REE analytically up to μ_0^4 -order for simple droplets. We draw our conclusion in section 4.

2 HEE of the mABJM Theory and LLM Geometries

Supersymmetric vacua of the mABJM theory are classified by the occupation numbers (N_n, N'_n) [9], which are numbers of irreducible $n \times (n+1)$ and $(n+1) \times n$ GRVV matrices [7], respectively. On the other hand, the LLM solutions with \mathbb{Z}_k orbifold are also classified by

the discrete torsions (l_n, l'_n) assigned in the droplet picture of the LLM geometry. It was shown that there exists one-to-one correspondence between (N_n, N'_n) and (l_n, l'_n) in the range, $0 \leq N_n, N'_n, l_n, l'_n \leq k$ [9]. Since the mass deformation of the ABJM theory is a relevant deformation from the UV fixed point, the dual LLM geometry with \mathbb{Z}_k orbifold is asymptotic to $\text{AdS}_4 \times S^7/\mathbb{Z}_k$. We investigate the behavior of the RG flow near the UV fixed point in terms of the HEE for all LLM solutions with general k and examine the F -theorem.

Let us start with the LLM geometry dual to the vacua of the $U(N)_k \times U(N)_{-k}$ mABJM theory with a mass parameter μ . The metric is given by

$$ds^2 = |G_{tt}|(-dt^2 + dw_1^2 + dw_2^2) + G_{xx}(dx^2 + dy^2) + G_{\theta\theta}ds_{S^3/\mathbb{Z}_k}^2 + G_{\tilde{\theta}\tilde{\theta}}ds_{\tilde{S}^3/\mathbb{Z}_k}^2, \quad (2.1)$$

where ds_{S^3/\mathbb{Z}_k}^2 and $ds_{\tilde{S}^3/\mathbb{Z}_k}^2$ are metrics for two S^3/\mathbb{Z}_k spheres and

$$G_{tt} = - \left(\frac{4\mu_0^2 y \sqrt{\frac{1}{4} - z^2}}{f^2} \right)^{2/3}, \quad G_{xx} = \left(\frac{f \sqrt{\frac{1}{4} - z^2}}{2\mu_0 y^2} \right)^{2/3}, \quad (2.2)$$

$$G_{\theta\theta} = \left(\frac{f y \sqrt{\frac{1}{2} + z}}{2\mu_0(\frac{1}{2} - z)} \right)^{2/3}, \quad G_{\tilde{\theta}\tilde{\theta}} = \left(\frac{f y \sqrt{\frac{1}{2} - z}}{2\mu_0(\frac{1}{2} + z)} \right)^{2/3} \quad (2.3)$$

with

$$f = \sqrt{1 - 4z^2 - 4y^2 V^2}. \quad (2.4)$$

In the metric, the mass parameter μ_0 is identified with that of the mABJM theory through $\mu_0 = \mu/4$ [9] in the convention of [11]. The geometry is completely determined by functions z and V ,

$$z(x, y) = \sum_{i=1}^{2N_B+1} \frac{(-1)^{i+1}(x - x_i)}{2\sqrt{(x - x_i)^2 + y^2}}, \quad V(x, y) = \sum_{i=1}^{2N_B+1} \frac{(-1)^{i+1}}{2\sqrt{(x - x_i)^2 + y^2}}, \quad (2.5)$$

where x_i 's denote the boundaries of black and white strips in the droplet representation with N_B being the number of black droplets. For details of the droplet picture with general k , see [9]. Due to the quantization condition of the four-form fluxes on 4-cycles ending on the edges of black/white regions, it turns out that x_i 's are quantized as

$$\frac{(x_{i+1} - x_i)}{2\pi l_P^3 \mu_0} \in \mathbb{Z}, \quad (2.6)$$

where l_P is the Planck length. Note that the quantization is proportional to μ_0 . It introduces μ_0 dependence to the metric in addition to the explicit dependence appearing in (2.2) and (2.3).

The metric (2.1) goes asymptotically to $AdS_4 \times S^7/\mathbb{Z}_k$ with radius R given by

$$R = (32\pi^2 k \tilde{N})^{1/6} l_P, \quad (2.7)$$

where¹

$$\begin{aligned} \tilde{N} &= \frac{1}{2k}(\tilde{C}_2 - \tilde{C}_1^2), \\ \tilde{C}_p &= \sum_{i=1}^{2N_B+1} (-1)^{i+1} \left(\frac{x_i}{2\pi l_P^3 \mu_0} \right)^p \equiv \sum_{i=1}^{2N_B+1} (-1)^{i+1} \tilde{x}_i^p. \end{aligned} \quad (2.8)$$

For later convenience, we define normalized coefficients C_p by

$$C_p \equiv (k\tilde{N})^{-p/2} \tilde{C}_p, \quad (2.9)$$

which are invariant under an overall scaling of x_i 's. Then $C_2 - C_1^2 = 2$.

Now, let's consider a 9 dimensional surface in this geometry. We denote coordinates of the surface by σ_i with $i = 1, \dots, 9$ and represent the embedding function as $X^M(\sigma^i)$ where $M = 0, \dots, 10$. Then, the 9 dimensional area of the surface becomes

$$\gamma_A = \int d^9\sigma \sqrt{\det g_{ij}} = \int d^9\sigma \sqrt{\det G_{MN} \frac{\partial X^M \partial X^N}{\partial \sigma^i \partial \sigma^j}}, \quad (2.10)$$

where g_{ij} is the induced metric of the surface and G_{MN} is the 11-dimensional metric in (2.1). The HEE is defined by [1, 2]

$$S_A = \frac{\text{Min}(\gamma_A)}{4G_N}, \quad (2.11)$$

where $G_N = (2\pi l_P)^9/(32\pi^2)$ is the 11 dimensional Newton constant. In the next section we would like to calculate S_A for disk-type entangling surfaces in the small μ_0 limit.

3 Anisotropic Minimal Surfaces and HEE

The effect of small mass deformation on the HEE has been considered in [11] under the approximation that the minimal surface is independent of the angle in polar coordinates introduced below. Though this approximation gives reasonable results consistent with F -theorem for simple droplet configurations, one cannot expect that it would be valid in general

¹ \tilde{N} is the area of the Young diagram made of positions \tilde{x}_i divided by k , and is equal to the rank N of the gauge group in the field theory up to the contribution of discrete torsions [28, 29]. See section 3 for an example. In addition, there is a further constant correction $-\frac{1}{24}(k - \frac{1}{k})$. With these corrections, \tilde{N} should eventually be the Maxwell M2 charge [9], i.e. $\tilde{N} = N - \frac{1}{24}(k - \frac{1}{k})$.

because the spherical symmetry is obviously broken. Indeed, for some droplet configurations the REE calculated in [11] does not decrease monotonically along the RG flow, violating the F -theorem. For these LLM geometries, it was also found that the curvature scalars are not small at some transverse positions even in the large N limit [11, 27], which implies that the gauge/gravity duality for those geometries does not work in this approximation. In this section we would like to investigate the effect of small mass deformation without resorting to such an approximation. In other words, we will treat the angular dependence of the minimal surface exactly. It amounts to solving PDEs with two variables up to some nontrivial order in μ_0 .

From now on, we take only disk type entangling surfaces into account. We will work with polar coordinates u and α defined by

$$x = \frac{R^3}{4lu} \cos \alpha, \quad y = \frac{R^3}{4lu} \sin \alpha, \quad (3.12)$$

where l is the radius of the disk at the boundary. The minimal surface is bounded by a disk in the w_1 - w_2 plane at the boundary of AdS space ($u = 0$). To describe such a configuration, we may consider the following embedding,

$$w_1 = \rho l \cos \sigma_1, \quad w_2 = \rho l \sin \sigma_1, \quad u = u(\rho, \phi), \quad \alpha = \alpha(\rho, \phi). \quad (3.13)$$

Plugging this into (2.10), we obtain the action,

$$\gamma_{\text{disk}} = \frac{\pi^5 R^9}{8k} \int_0^1 d\rho \int_0^\pi d\phi \frac{g\rho \sin^3 \alpha}{u^2} \sqrt{\alpha'^2 + \frac{u'^2}{u^2} + g^2(\dot{\alpha}u' - \alpha'\dot{u})^2}, \quad (3.14)$$

where $\dot{} = \frac{\partial}{\partial \rho}$ and $' = \frac{\partial}{\partial \phi}$. We have also introduced the function $g(u, \phi)$ defined by

$$f(u, \phi) = 2\mu_0 l u \sin \phi g(u, \phi). \quad (3.15)$$

Note that all the mass-deformation effect in (3.14) appears only through the function g which contains the information of the droplet position x_i 's. In the undeformed limit $\mu_0 \rightarrow 0$, x_i 's go to zero due to the quantization condition (2.6). Then it is easy to see that g goes to unity and hence (3.14) reduces to the minimal surface action for the undeformed ABJM theory [1].

By utilizing the residual gauge degree of freedom, we may choose $\alpha = \phi$. Then the equation of motion yields a PDE for $u = u(\rho, \phi)$,

$$\frac{\partial}{\partial \rho} \left(\frac{\rho g^3 \dot{u} \sin^3 \phi}{u^2 \sqrt{\mathcal{X}}} \right) + \frac{\partial}{\partial \phi} \left(\frac{\rho g u' \sin^3 \phi}{u^4 \sqrt{\mathcal{X}}} \right) - \frac{\partial}{\partial u} \left(\frac{\rho g \sin^3 \phi}{u^2} \sqrt{\mathcal{X}} \right) = 0, \quad (3.16)$$

where $\mathcal{X} = 1 + \frac{u'^2}{u^2} + g^2 \dot{u}^2$.

3.1 HEE up to $\mathcal{O}(\mu_0^2)$

Now we are ready to consider the effect of small mass deformation. From (2.4) and (2.5) we see that g can be expanded in powers of $\frac{x_i}{\sqrt{x^2+y^2}} \sim \mu_0 l u$. Furthermore, u itself will depend on $\mu_0 l$. As a result we can write

$$\begin{aligned} u &\equiv \sum_{n=0}^{\infty} u_n(\rho, \phi) (\mu_0 l)^n = u_0(\rho) + u_1(\rho, \phi) \mu_0 l + u_2(\rho, \phi) (\mu_0 l)^2 + \dots, \\ g(u, \phi) &\equiv \sum_{n=0}^{\infty} g_n(\phi) (u \mu_0 l)^n \\ &= 1 + g_1(\phi) u_0 \mu_0 l + [g_1(\phi) u_1 + g_2(\phi) u_0^2] (\mu_0 l)^2 + \dots. \end{aligned} \quad (3.17)$$

Since z and V consist of the generating function of the Legendre polynomials, g_i 's can be written in terms of Legendre polynomials [11]. Explicitly, a few lower terms are

$$\begin{aligned} g_1(\phi) &= D_1 \cos \phi, \\ g_2(\phi) &= D_2 + D_3 \cos(2\phi), \end{aligned} \quad (3.18)$$

where D_i 's are constants depending on the droplet positions,

$$\begin{aligned} D_1 &= \frac{1}{\sqrt{2}} (C_3 - C_1 C_2), \\ D_2 &= -\frac{1}{32} [4C_3^2 - 4C_1(C_1^2 + C_2)C_3 - C_2^2(C_1^2 - 5C_2) + 9(C_1^2 - C_2)C_4], \\ D_3 &= -\frac{1}{32} [4C_3^2 - 4C_1(3C_1^2 - C_2)C_3 + C_2^2(C_1^2 + 3C_2) + 15(C_1^2 - C_2)C_4], \end{aligned} \quad (3.19)$$

and C_p 's are defined in (2.9). Given D_i 's, one can solve the equations of motion (3.16) perturbatively with respect to μ_0 to obtain the change of the minimal surface. Note that we have to solve inhomogeneous PDEs of two variables ρ and ϕ in the background of lower order configurations. There is, in general, no guarantee that explicit form of solutions can be obtained. Nevertheless, in this case, we are able to find exact solutions up to the nonvanishing second orders in perturbation.

Let us start with the zeroth order equation of (3.16) in μ_0 ,

$$\ddot{u}_0 + \frac{(2\rho + u_0 \dot{u}_0)(1 + \dot{u}_0^2)}{\rho u_0} = 0. \quad (3.20)$$

This is nothing but the equation of motion for the conformal case, as it should be. Imposing the boundary conditions, $u(0) = 1$ and $\dot{u}(0) = 0$, one can find the well-known solution which is a geodesic in AdS space,

$$u_0(\rho) = \sqrt{1 - \rho^2}. \quad (3.21)$$

This gives the minimal surface for ABJM theory without mass deformation.

If we plug this solution into (3.16), then the first order equation of motion reads

$$\begin{aligned} & \rho(1 - \rho^2)^2 \ddot{u}_1 + \rho u_1'' + (1 - \rho^2)(1 - 2\rho^2) \dot{u}_1 + 3\rho \cot \phi u_1' - 2\rho u_1 \\ & - D_1 \rho(1 - \rho^2)(5 - 3\rho^2) \cos \phi = 0. \end{aligned} \quad (3.22)$$

We have to solve the equation under the boundary conditions $u_1(1, \phi) = \dot{u}_1(0, \phi) = 0$ and $u_1'(\rho, 0) = u_1'(\rho, \pi) = 0$, where the latter comes from the regularity at $\phi = 0$ and π . This is an inhomogeneous linear PDE with explicit dependence on the independent variables ρ and ϕ . It, however, admits a very simple solution

$$u_1(\rho, \phi) = -\frac{D_1}{2} (1 - \rho^2) \cos \phi. \quad (3.23)$$

One can proceed to the second order in μ_0^2 . The equation of motion at the second order becomes, after the solutions (3.21) and (3.23) plugged into (3.17),

$$\begin{aligned} & \rho(1 - \rho^2)^2 \ddot{u}_2 + (1 - \rho^2)(1 - 2\rho^2) \dot{u}_2 + \rho(u_2'' + 3 \cot \phi u_2') - 2\rho u_2 \\ & + \frac{1}{8} \rho(1 - \rho^2)^{3/2} \left[D_1^2(27 - 26\rho^2) - 16D_2(3 - 2\rho^2) \right. \\ & \left. + (11D_1^2 - 16D_3)(3 - 2\rho^2) \cos(2\phi) \right] = 0, \end{aligned} \quad (3.24)$$

with the boundary conditions $u_2'(\rho, 0) = u_2'(\rho, \pi) = 0$ and $u_2(1, \phi) = \dot{u}_2(0, \phi) = 0$. This is even more complicated than the first order equation (3.22). Remarkably, however, a fully analytic solution is still available,

$$\begin{aligned} u_2(\rho, \phi) = & -\frac{1}{6\sqrt{1 - \rho^2}} (D_1^2 + 20D_2 - 12D_3) \log(1 + \sqrt{1 - \rho^2}) \\ & + \frac{1}{48} \left\{ [8 + (9 - 13\rho^2)\sqrt{1 - \rho^2}] D_1^2 + 16[10 - (6 - \rho^2)\sqrt{1 - \rho^2}] D_2 \right. \\ & \left. - 48(2 - \sqrt{1 - \rho^2}) D_3 \right\} + \frac{1}{48} (11D_1^2 - 16D_3)(1 - \rho^2)^{3/2} \cos(2\phi). \end{aligned} \quad (3.25)$$

Having found the solution to the μ_0^2 -order, now we can calculate the minimal surface area (3.14) to this order,

$$\gamma_{\text{disk}} = \gamma_{\text{disk}}^{(0)} + \mu_0 l \gamma_{\text{disk}}^{(1)} + (\mu_0 l)^2 \gamma_{\text{disk}}^{(2)} + \dots \quad (3.26)$$

Inserting the solutions (3.21), (3.23) and (3.25) into (3.14), we obtain²:

$$\gamma_{\text{disk}}^{(0)} = \frac{\pi^5 R^9}{6k} \left(\frac{l}{\epsilon} - 1 \right), \quad \gamma_{\text{disk}}^{(1)} = 0, \quad \gamma_{\text{disk}}^{(2)} = -\frac{\pi^5 R^9}{72k} (12D_3 - D_1^2 - 20D_2), \quad (3.27)$$

² Introducing the UV cutoff ϵ in the u coordinate changes the upper limit of the integration range of ρ in (3.14).

where ϵ in $\gamma_{disk}^{(0)}$ is the UV cutoff in the u coordinate. Note that the first order correction vanishes due to the angular integration. For the second order contribution $\gamma_{disk}^{(2)}$, it is crucial to notice that the combination $(12D_3 - D_1^2 - 20D_2)$ can be rewritten in the form of a complete square,

$$12D_3 - D_1^2 - 20D_2 = 16 + \frac{1}{2}(C_3 - 3C_1C_2 + 2C_1^3)^2, \quad (3.28)$$

where we used the parameter relations in (3.19). The entanglement entropy then becomes

$$S_{\text{disk}} = \frac{\pi^5 R^9}{24G_N k} \left\{ \frac{l}{\epsilon} - 1 - \mu_0^2 l^2 \left[\frac{4}{3} + \frac{1}{24}(C_3 - 3C_1C_2 + 2C_1^3)^2 \right] \right\} + \mathcal{O}(\mu_0^3). \quad (3.29)$$

A few comments are in order. First, it is not difficult to show that the expression $(C_3 - 3C_1C_2 + 2C_1^3)$ appearing here is a unique combination made of cubic terms in x_i which is invariant under the translation $x_i \rightarrow x_i + a$. It provides a nontrivial consistency check of the result. Moreover, the expression appears in (3.29) in the form of a complete square and hence the second order term is negative definite. This has an important implication in relation to the F -theorem [22–24]. In the present context, the REE can play the role of a c -function of the theory [10]. It is computed by the prescription

$$\begin{aligned} \mathcal{F}_{\text{disk}} &\equiv \left(l \frac{\partial}{\partial l} - 1 \right) S_{\text{disk}} \\ &= \frac{\pi^5 R^9}{24G_N k} \left\{ 1 - \mu_0^2 l^2 \left[\frac{4}{3} + \frac{1}{24}(C_3 - 3C_1C_2 + 2C_1^3)^2 \right] \right\} + \mathcal{O}(\mu_0^3), \end{aligned} \quad (3.30)$$

which is clearly monotonically decreasing near the UV fixed point. Note that this is true for *all* geometries dual to the vacua of mABJM, regardless of the C_i 's. This result may be considered as an evidence of the validity of holography for non-conformal case. This corrects the result in [11] where REE was calculated with the angle (α) dependence neglected and showed increasing behavior for some asymmetric droplet configurations. This means that the angle dependence of the minimal surface in the LLM geometry results in nontrivial contributions.

As is evident from the calculation, the REE depends on N or k only through C_p 's except the overall multiplication. Therefore, different theories with same C_p 's will give essentially the same REE. Since C_p 's defined in (2.9) are invariant under the overall scaling of droplet boundaries x_i 's, a family of droplets with a same geometric shape of Young diagrams modulo overall size give the same REE up to a multiplication factor. This holds to all orders in μ_0 since the action (3.14) is completely determined in terms of C_p 's up to an overall factor.

As a simple example, we consider the case of $N_B = 1$ with arbitrary k for which the geometry is specified by three boundary positions x_1 , x_2 and x_3 of the droplet. See Fig.1.

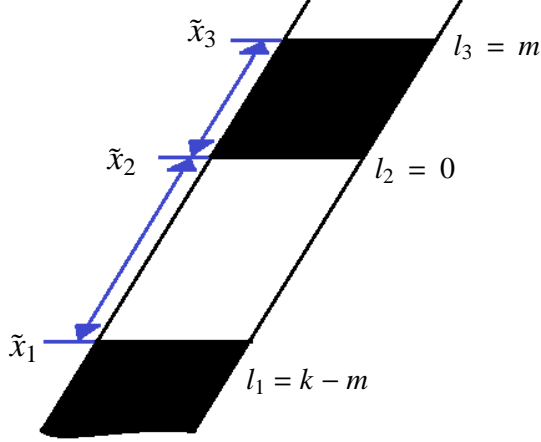


Figure 1: A droplet representation of $N_B=1$ case. l_i 's are discrete torsions assigned at \tilde{x}_i 's.

From (2.8), the REE becomes

$$\mathcal{F}_{\text{disk}} = \frac{\pi^5 R^9}{24 G_N k} \left\{ 1 - \mu_0^2 l^2 \left[\frac{7}{12} + \frac{3}{8} \left(\frac{x_3 - x_2}{x_2 - x_1} + \frac{x_2 - x_1}{x_3 - x_2} \right) \right] \right\} + \mathcal{O}(\mu_0^3). \quad (3.31)$$

This result explicitly shows that scaling the overall size of the droplet (or the shape of the Young diagram) does not change REE except the overall multiplication factor.

To connect the result with the field theory, let us parameterize the integer-valued positions $\tilde{x}_i = x_i / 2\pi l_P^3 \mu_0$ defined in (2.8) as $\tilde{x}_1 = -pk - m$, $\tilde{x}_2 = (q - p)k$, $\tilde{x}_3 = qk + m$, where p, q, m are positive integers and $0 \leq m < k$, so that the location of the Fermi level of the droplet is zero. Then including the contribution of the discrete torsions [9], we obtain the rank N of the gauge group as $N = (pk + m)(qk + m)/k - m(m - k)/k = kpq + m(p + q + 1)$. In the large N limit with finite 't Hooft coupling $\lambda = N/k$, (3.31) is reduced to

$$\mathcal{F}_{\text{disk}} = \frac{\pi^5 R^9}{24 G_N k} \left\{ 1 - \mu_0^2 l^2 \left[\frac{7}{12} + \frac{3}{8} \left(\frac{p}{q} + \frac{q}{p} \right) \right] + \mathcal{O}(1/k) \right\} + \mathcal{O}(\mu_0^3), \quad (3.32)$$

where we assumed $k \gg m$ for simplicity and $\lambda = pq$ in this limit. Therefore, for a fixed ratio of p and q the mass correction is independent of λ . On the other hand, if we scale, say, p with q fixed, then the correction depends nontrivially on λ since $p/q = \lambda/q^2$. Note that by changing p or q , we are comparing theories with different λ . The above result demonstrates that λ -dependence of REE appears in different ways, depending on how vacua are selected in mABJM theories for comparison.

Finally, let us give a comment on the stationarity of the RG flow. Since the deformation parameter μ enters into the mABJM theory as a mass of a supermultiplet, the deformation

is of the first order in μ due to the fermionic mass term. On the other hand, REE \mathcal{F} in (3.30) has vanishing first order correction in μ_0 . It means that the RG flow at UV point is stationary. This is consistent with the result of [13].

3.2 HEE up to $\mathcal{O}(\mu_0^4)$ for symmetric droplet case

For some simple droplet configurations, it is possible to obtain higher order corrections analytically. For example, if $D_1 = 0$, the first order equation (3.22) becomes homogeneous with vanishing boundary conditions and hence the first order correction u_1 is zero identically. This will also simplify higher order equations. Here we will consider a symmetric droplet case obtained by putting $p = q$ in Fig.1, i.e., $(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3) = (-pk - m, 0, pk + m)$ and $N = kp^2 + m(2p + 1)$.

To consider higher order corrections up to μ_0^4 -order, we need the following coefficient functions in (3.17),

$$\begin{aligned} g_2(\phi) &= -\frac{1}{8} + \frac{9}{8} \cos(2\phi), \\ g_4(\phi) &= \frac{1}{256} [85 - 60 \cos(2\phi) + 359 \cos(4\phi)], \end{aligned} \quad (3.33)$$

as well as $g_n(\phi) = 0$ for odd n . By symmetry we may set $u_n = 0$ for odd n . Then employing all the lower order results including (3.21) and (3.25) gives us the equation of motion for $u_4(\rho, \phi)$,

$$\begin{aligned} \rho(1 - \rho^2)^2 \ddot{u}_4 + (1 - \rho^2)(1 - 2\rho^2) \dot{u}_4 + \rho(u_4'' + 3 \cot \phi u_4') - 2\rho u_4 \\ + B_0^{(4)}(\rho) + B_2^{(4)}(\rho) \cos(2\phi) + B_4^{(4)}(\rho) \cos(4\phi) = 0, \end{aligned} \quad (3.34)$$

where

$$\begin{aligned} B_0^{(4)}(\rho) &= \frac{64\rho(3\rho^2 + 2)}{9(1 - \rho^2)^{3/2}} \left[\log \left(\sqrt{1 - \rho^2} + 1 \right) \right]^2 - \frac{2}{9\rho} (9\rho^6 + 28\rho^4 + 187\rho^2 - 64) \\ &\quad - \frac{2\rho}{9(1 - \rho^2)} \left[128\rho^2 - (67\rho^4 + 8\rho^2 + 85)\sqrt{1 - \rho^2} + 192 \right] \log \left(\sqrt{1 - \rho^2} + 1 \right) \\ &\quad + \frac{1}{1152\rho\sqrt{1 - \rho^2}} (6147\rho^{10} - 15629\rho^8 + 6165\rho^6 - 8463\rho^4 + 69124\rho^2 - 16384), \\ B_2^{(4)}(\rho) &= \frac{3}{32} \rho(1 - \rho^2) \left[576 - 192\rho^2 - (15\rho^4 - 227\rho^2 + 404)\sqrt{1 - \rho^2} \right] \\ &\quad - 6\rho(7 - \rho^2)\sqrt{1 - \rho^2} \log \left(\sqrt{1 - \rho^2} + 1 \right), \\ B_4^{(4)}(\rho) &= -\frac{125}{128} \rho(1 - \rho^2)^{5/2} (4 - 3\rho^2). \end{aligned} \quad (3.35)$$

This is again an inhomogeneous PDE with very complicated source terms but fortunately we can find the solution satisfying the necessary boundary conditions,

$$u_4(\rho, \phi) = C_4^{(4)}(\rho) \cos(4\phi) + C_2^{(4)}(\rho) \cos(2\phi) + C_0^{(4)}(\rho), \quad (3.36)$$

where

$$\begin{aligned} C_4^{(4)}(\rho) &= -\frac{25}{256} (1 - \rho^2)^{5/2}, \\ C_2^{(4)}(\rho) &= \frac{1}{64 (1 - \rho^2)^{3/2}} \left[96 (7 - 2\rho^2) \rho^2 \log \left(\sqrt{1 - \rho^2} + 1 \right) + 32 (6\rho^4 - 16\rho^2 - 5) \sqrt{1 - \rho^2} \right. \\ &\quad \left. + (1 - \rho^2) (3\rho^6 - 97\rho^4 + 209\rho^2 + 125) \right], \\ C_0^{(4)}(\rho) &= -\frac{1}{\sqrt{1 - \rho^2}} \int_\rho^1 F(s) ds, \end{aligned} \quad (3.37)$$

with

$$\begin{aligned} F(s) &= \frac{135s^6 + 598s^4 - 3371s^2 + 1423}{135s(1 - s^2)^{3/2}} - \frac{256}{9s} \sqrt{1 - s^2} \log 2 \\ &\quad + \frac{18441s^8 - 37339s^6 - 170949s^4 + 326871s^2 - 182144}{17280s(1 - s^2)} - \frac{64s \log^2(\sqrt{1 - s^2} + 1)}{9(1 - s^2)^2} \\ &\quad + \frac{1}{9s} \left[\frac{64(s^4 + 1)}{(1 - s^2)^{3/2}} - \frac{134s^6 - 524s^4 + 501s^2 - 192}{(1 - s^2)^2} \right] \log(\sqrt{1 - s^2} + 1). \end{aligned} \quad (3.38)$$

Inserting the solutions in (3.21), (3.25), and (3.36) into (3.17) and (3.14), we easily obtain the corresponding HEE and REE for the symmetric droplet up to μ_0^4 -order,

$$\begin{aligned} S_{\text{disk}} &= \frac{\pi^5 R^9}{24kG_N} \left(\frac{l}{\epsilon} - 1 - \frac{4}{3} \mu_0^2 l^2 + \frac{2671 - 3840 \log 2}{540} \mu_0^4 l^4 \right) + \mathcal{O}(\mu_0^6), \\ \mathcal{F}_{\text{disk}} &= \frac{\pi^5 R^9}{24kG_N} \left(1 - \frac{4}{3} \mu_0^2 l^2 + \frac{2671 - 3840 \log 2}{180} \mu_0^4 l^4 \right) + \mathcal{O}(\mu_0^6). \end{aligned} \quad (3.39)$$

This result holds for arbitrary N and k . In general, the expressions of the HEE and REE depend on N , k , radius of the disk, and the choice of droplets as we have seen in the previous subsection. However, for a family of droplets related by rescaling of the overall size, the coefficients of the corrections are given by pure numbers as seen in (3.39), and in particular are independent of the 't Hooft coupling constant λ . This is an interesting phenomena from the point of view of the gauge/gravity duality. Based on the duality relation between the vacua of the mABJM theory and the LLM geometries with \mathbb{Z}_k orbifold, one can examine the HEE conjecture by computing the EE of the dual field theory on a family of vacua considered here.

4 Conclusion

We investigated the RG flow behavior and the F -theorem in terms of the HEE near the UV fixed point in 3-dimensions, where a supersymmetric Chern-Simons matter theory is living. As the UV CFT we considered the $\mathcal{N} = 6$ ABJM theory and introduced the supersymmetry preserving mass deformation, called the mABJM theory. This deformation is a relevant deformation and so triggers the RG flow from the UV fixed point. To describe the RG behavior near the UV fixed point, we adapted the HEE conjecture to the LLM geometry in 11-dimensional supergravity, since the supersymmetric vacua of the mABJM theory have one-to-one correspondence with the LLM geometries with \mathbb{Z}_k orbifold.

The LLM solution has $\text{SO}(2,1) \times \text{SO}(4) \times \text{SO}(4)$ isometry and warp factors of the metric depend on the two transverse coordinates (u, α) in (3.12). For this reason, one has to solve the PDE for u and α to obtain the minimal surface in the HEE proposal. In this paper, we analytically solved the PDE up to μ_0^2 -order for all LLM solutions with arbitrary N and k . We found that REEs have different RG trajectories depending on the LLM geometries but they are always monotonically decreasing near the UV fixed point in accordance with the F -theorem in 3-dimensions.³ We also computed the REE up to μ_0^4 -order for a special family of LLM geometries with arbitrary N and k . It would be interesting to extend the result to more general case.

Since the HEE proposal is based on the gauge/gravity duality, in order to compare our results in gravity side with those in the mABJM theory, one has to consider the large 't Hooft coupling $\lambda = N/k$ in the $N \rightarrow \infty$ limit. In general, the effect of mass deformation in REE would depend on λ and calculation in field theory side is a formidable task due to nonperturbative effects coming from the strong coupling constant. In the mABJM theory, there are further complications from the presence of many vacua. However, we found that for a family of droplets with a same shape of Young diagrams, we have the same REE (or HEE) up to overall dependence on N and k . One might be able to calculate the REE in the field theory side perturbatively on a certain class of vacua, and compare the result quantitatively with that in the dual gravity side.

³Recently, it was reported that the strong form of the F -theorem, which describes the monotonically decreasing behavior of the REE, is violated for relevant deformations by operators of conformal dimension $3/2 < \Delta < 5/2$ [30, 31]. One needs more investigations in this direction.

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